Chapter 1

Elasticity and Thermal Stress

1.1 Elastic and Plastic deformation

When a force is applied to an object, it deforms. Of course, some objects are deformed a lot, and some objects are so deformed that they are barely recognizable. The amount of this deformation is a characteristic property of an object. Elastic deformation means that an object is deformed while a force is applied and returns to its original shape when the force is removed. Therefore, this deformation is sometimes referred to as a "reversible deformation". If a lot of external force is applied, it may not return to its original shape even if the force is removed. In this case, we call this plastic deformation. You can think of elasticity theory as an academic field that deals with the elastic deformation of objects. It is a concept that I think you have already encountered in previous classes, such as solid mechanics.

1.2 Stresses and Strains

Fig. 1.1 shows a cross-sectional area rod A acting by force F. All planes perpendicular to the rod's axis experience the same force. The stress in place $a - a'$ is

$$\sigma_n = \frac{F}{A}$$

The stress is called normal (subscript n) if it acts in the direction perpendicular to the stressed plane. The stress that tends to pull the atomic planes apart is termed tensile. If the force F is applied in opposite directions, the solid will be squeezed together. This stress is called compressive. By convention, tensile stresses are positive and compressive stresses are negative. In Fig. 1.2, the force F is applied on the top plane in the y direction. The stress is generated parallel to the surface rather than perpendicular to the surface. This stress is termed a shear stress (subscript s).

$$\sigma_s = \frac{F}{A}$$

If the applied load (force) is not purely normal or purely shear, arbitrarily oriented planes in the body will experience both normal and shear stress components.

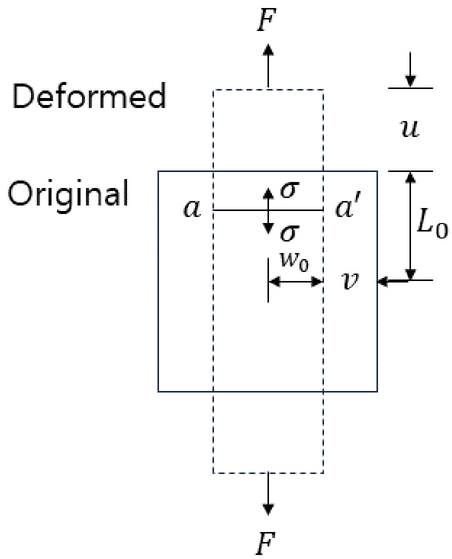

Figure 1.1: Normal stress

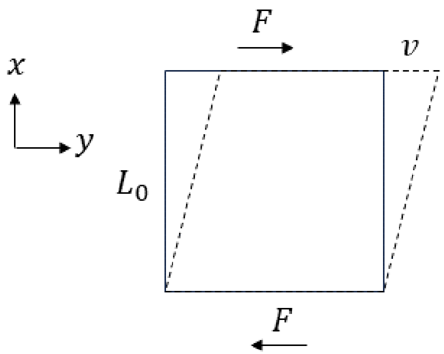

Figure 1.2: Shear stress

1.2.1 Notation for stresses

In general, any point within a solid subjected to one or more loads may have up to six stress components. Notation σ_n or σ_s is a simplified version. In principle, we represent stress by σ_{ij} where i is the plane on which the stress component acts and j is its direction. i and j can be x, y, z in cartesian coordinate and r, θ, z in cylindrical coordinate, r, θ, ϕ in spherical coordinate. The stress indicated σ in Fig. 1.1 is σ_{xx}; it acts on the $y - z$ plane.

All normal stresses bear the generic designation σ_{ii}, which is often shortened to σ_i. In three dimensions, there are at most three non-zero normal components of the stress at a point.

The proper designation of the shear stress components cannot be reduced to a single subscript because i and j in σ_{ij} are always different.

1.2.2 Displacements and strains

Displacements are changes in the position of a point in a body between the unstressed and stressed states. The strain is a fractional displacement. In common with stresses, displacements and strains come in two varieties, normal and shear.

Fig. 1.1 shows the normal displacements u and v in the x and y directions. The original dimensions of the piece are L_o and w_o. The solid rectangle represents the stress-free solid, and the dashed rectangle represents its shape following application of the axial force. The displacement u (positive) corresponds to the outward movement of the horizontal surfaces, and the displacement v (negative) represents the shrinkage of the sides of the body.

The strains in the x and y directions are defined as fractional displacements:

$$\varepsilon_x = \frac{u}{L_0} \qquad \varepsilon_y = \frac{v}{w_0}$$

In Fig. 1.2, a shear displacement v resulting from a force F acting on the side of the body at a distance L_0 from the fixed bottom. The shear strain is defined as the ratio of these two lengths:

$$\varepsilon_{xy} = \frac{v}{L_0}$$

1.2.3 Generalization of strain definition

The stress notation can be generalized for stress analyzes.

Cylindrical coordinate

In cartesian coordinate, we can have the position vector by

$$\vec{s} = x\hat{i} + y\hat{j} + z\hat{k}$$

the position vector can be represented by

$$\vec{s} = r\cos\theta\hat{i} + r\sin\theta\hat{j} + z\hat{k}$$

in Fig.1.3. The unit vector along the radial direction in cylindrical coordinate is given by $\hat{r}$,

$$\hat{r} = \frac{\partial\vec{s}}{\partial r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

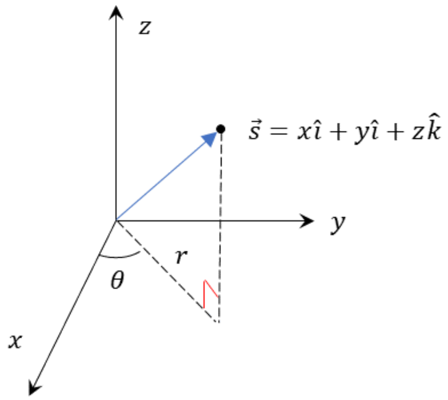

Figure 1.3: Position vector of $\vec{s}(x, y, z)$ and quantity r and θ.

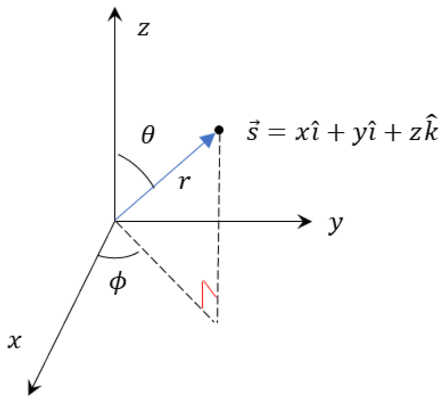

Figure 1.4: Position vector of $\vec{s}(x, y, z)$ and quantity r, θ and ϕ.

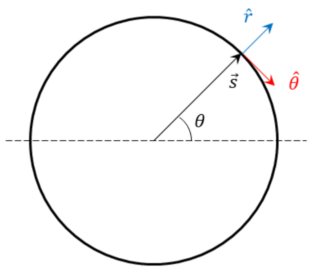

Figure 1.5: The schematic plot of $\hat{r}$ and $\hat{\theta}$

the position vector along θ direction is

$$\vec{\theta} = \frac{\partial \vec{s}}{\partial \theta} = -r\sin\theta\hat{i} + r\cos\theta\hat{j}$$

therefore, the unit position vector along θ direction is

$$\hat{\theta} = \frac{1}{r}\frac{\partial \vec{s}}{\partial \theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

The schematic plots of $\hat{r}$ and $\hat{\theta}$ are visualized in Fig. 1.5. In spherical coordinate,

$$\vec{s} = r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}$$

The unit vector along radial direction in spherical coordinate is

$$\hat{r} = \frac{\partial \vec{s}}{\partial r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

Also,

$$\hat{\theta} = \frac{1}{r}\frac{\partial \vec{s}}{\partial \theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

and

$$\hat{\phi} = \frac{1}{r\sin\theta}\frac{\partial \vec{s}}{\partial \phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

The derivative is

$$\frac{\partial \hat{r}}{\partial \theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k} = \hat{\theta} \tag{1.1}$$

$$\frac{\partial \hat{r}}{\partial \phi} = -\sin\theta\sin\phi\hat{i} + \sin\theta\cos\phi\hat{j} = \sin\theta\hat{\phi} \tag{1.2}$$

Del operator

The differential operator in three-dimensional system in cartesian coordinate is

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

in cylindrical coordinate is

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{k} \tag{1.3}$$

in spherical coordinate is

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi} \tag{1.4}$$

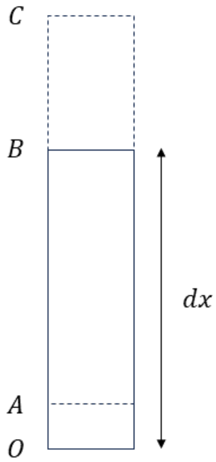

Figure 1.6: Definition of Normal stress

Normal strain

In Fig. 1.6 shows a deformation in the x direction. The original length of the body OB is taken to be a differential element dx. Upon application of a normal stress, the displacement of the bottom and top surfaces is

$$OA = u_x \qquad\qquad BC = u_x + \left(\frac{\partial u_x}{\partial x}\right) dx$$

The strain is

$$\frac{BC - OA}{OB} = \frac{\partial u_x}{\partial x} = \varepsilon_{xx} = \varepsilon_x$$

Consistently, we can define

$$\varepsilon_{yy} = \varepsilon_y = \frac{\partial u_y}{\partial y} \qquad\qquad \varepsilon_{zz} = \varepsilon_z = \frac{\partial u_z}{\partial z}$$

in cylindrical coordinate, assume only the displacement along r direction

$$\mathbf{u} = u_r(r,\theta)\hat{r}$$

$$\varepsilon_{rr} = \varepsilon_r = \left|\frac{\partial \mathbf{u}}{\partial r}\right| = \frac{\partial u_r(r,\theta)}{\partial r}$$

the hoop strain, with Eq. 1.1, $\varepsilon_{\theta\theta} = \varepsilon_\theta$ is

$$\varepsilon_{\theta\theta} = \varepsilon_\theta = \frac{1}{r}\left|\frac{\partial \mathbf{u}}{\partial \theta}\right| = \frac{1}{r}\frac{\partial u_r(r,\theta)}{\partial \theta} + \frac{u_r(r,\theta)}{r}$$

When radial displacement, $u_r(r,\theta)$ does not have θ dependence, i.e. has polar symmetry, we have

$$\varepsilon_{\theta\theta} = \frac{u_r(r)}{r} \tag{1.5}$$

For spherical geometry, with spherical symmetry, with Eqs. 1.1 and 1.2,

$$\varepsilon_r = \frac{\partial u_r(r)}{\partial r}$$

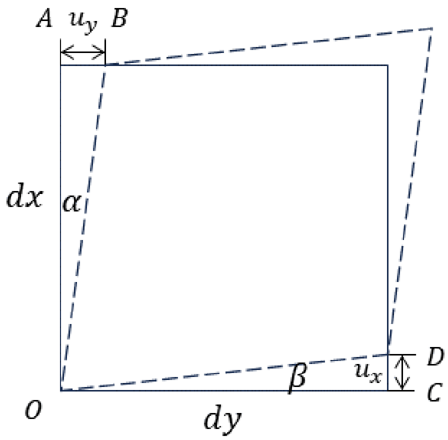

Figure 1.7: Definition of Shear stress

$$\varepsilon_\theta = \frac{1}{r}\left|\frac{\partial \mathbf{u}}{\partial \theta}\right| = \frac{u_r(r)}{r}$$

$$\varepsilon_\phi = \frac{1}{r\sin\phi}\left|\frac{\partial \mathbf{u}}{\partial \phi}\right| = \cancel{\frac{\partial u_r(r)}{\partial \phi}} + \frac{u_r(r)}{r} = \frac{u_r(r)}{r}$$

where $u_r(r)$ is the radial displacement, ϕ is the polar angle, and θ is the azimuthal angle.

Shear strain

In Fig. 1.7, the lower left corners of the original and deformed figure are superimposed at the point O. We have

$$u_y = AB = \left(\frac{\partial u_y}{\partial x}\right)dx$$

Proceed to

$$\tan\alpha \simeq \alpha = \frac{AB}{dx} = \frac{\partial u_y}{\partial x}$$

$$\tan\beta \simeq \beta = \frac{CD}{dy} = \frac{\partial u_x}{\partial y}$$

Therefore, the shear strain is

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\alpha+\beta}{2} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]$$

1.3 Equilibrium conditions

In two-dimensional system, the net force along x direction, the force on the system

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx - \sigma_{xx}\right)dy + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy - \sigma_{yx}\right)dx = 0$$

It proceed to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = \nabla \cdot \sigma = 0$$

In axisymmetric cylindrical coordinates, the radial equilibrium condition is

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial\sigma_{rz}}{\partial z} = 0 \tag{1.6}$$

and for the z direction it is

$$\frac{1}{r}\frac{\partial(r\sigma_{rz})}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} = 0$$

In spherical coordinates with spherical symmetry, the radial equilibrium condition is

$$\frac{\partial\sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

and by symmetry,

$$\sigma_{\theta\theta} = \sigma_{\phi\phi}$$

1.4 Stress-Strain Relations

For isotropic materials, the Young's modulus E is given by

$$E = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{\sigma_{yy}}{\varepsilon_{yy}} = \frac{\sigma_{zz}}{\varepsilon_{zz}}$$

When more than one normal stresses act on a body, the Poisson's ratio is given by, assume the positive strain is applied along x direction,

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}}$$

The relations between stress-strain are

$$\varepsilon_{xx} = \frac{1}{E}\Big[\sigma_{xx} - \nu\big(\sigma_{yy} + \sigma_{zz}\big)\Big] \tag{1.7a}$$

$$\varepsilon_{yy} = \frac{1}{E}\Big[\sigma_{yy} - \nu\big(\sigma_{xx} + \sigma_{zz}\big)\Big] \tag{1.7b}$$

$$\varepsilon_{zz} = \frac{1}{E}\Big[\sigma_{zz} - \nu\big(\sigma_{xx} + \sigma_{yy}\big)\Big] \tag{1.7c}$$

For shear stresses and strains

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{G} \qquad \varepsilon_{xz} = \frac{\sigma_{xz}}{G} \qquad \varepsilon_{yz} = \frac{\sigma_{yz}}{G}$$

where G is the shear modulus. From Eqs. 1.7a to 1.7c, we have

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1-2\nu}{E}\big(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\big) \tag{1.8}$$

For shear stresses and strains

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{G} \qquad \varepsilon_{xz} = \frac{\sigma_{xz}}{G} \qquad \varepsilon_{yz} = \frac{\sigma_{yz}}{G}$$

where G is the shear modulus. Assume the pure shear state,

$$\sigma_{xx} = \sigma \qquad \sigma_{yy} = -\sigma \qquad \sigma_{zz} = 0$$

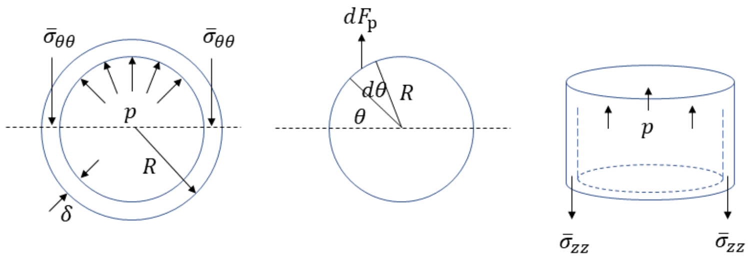

Figure 1.8: Average stresses in a thin-wall cylinder.

then we have

$$\varepsilon_{xx} = \frac{\sigma}{E}(1+\nu) \qquad \varepsilon_{yy} = -\frac{\sigma}{E}(1+\nu) \qquad \varepsilon_{zz} = 0$$

The shear strain is

$$\varepsilon_{xy} = \gamma_{\max} = 2\varepsilon_{xx} = \frac{2\sigma}{E}(1+\nu)$$

The shear modulus is

$$G = \frac{\sigma_{xy}}{\varepsilon_{xy}} = \frac{\sigma}{\varepsilon_{xy}} = \frac{E}{2(1+\nu)}$$

The mean hydrostatic stress σ_h is

$$\sigma_h = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

The volume after deformation is V_f and the volume before deformation is V, then

$$\frac{V_f}{V} = 1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

1.5 Thin-wall cylinders

If the radius of a hollow cylinder is much larger than thickness of the wall, we can apply thin-wall approximation. In Fig. 1.8, a cross section of a thin-wall cylinder of radius R and wall thickness δ that is internally stressed by a pressure p. The pressure acts radially on the inner surface and so must be resolved in the vertical direction to balance the azimuthal stress acting on the midplane section of the tube wall. The force

$$F_p = \int dF_p = pR \int_o^\pi \sin\theta d\theta = 2pR$$

At equilibrium, F_p is opposed by the hoop(circumferential) stress on the area δ on both sides of the cross section.

$$F_s = 2\delta\overline{\sigma}_{\theta\theta}$$

Since

$$F_p = F_s$$

it proceeds to

$$2pR = 2\delta\overline{\sigma}_{\theta\theta} \rightarrow \overline{\sigma}_{\theta\theta} = \frac{pR}{\delta}$$

$\overline{\sigma}_{\theta\theta}$ is called the hoop or circumferential stress. The force exerted on the closed top

$$\pi R^2 p$$

is balanced by the tensile force in the thin wall,

$$2\pi R\delta\overline{\sigma}_{zz}$$

the axial stress is

$$\overline{\sigma}_{zz} = \frac{pR}{2\delta} \tag{1.9}$$

it is easily shown that the hoop stress is twice greater than the axial stress.

1.6 Thick-wall cylinders

From Eq. 1.5,

$$\frac{d\varepsilon_{\theta\theta}}{dr} = \frac{d}{dr}\left(\frac{u_r}{r}\right) = \frac{1}{r}\left(\frac{du_r}{dr} - \frac{u_r}{r}\right) = \frac{\varepsilon_{rr} - \varepsilon_{\theta\theta}}{r} \tag{1.10}$$

From Eqs. 1.7a to 1.7c,

$$E\frac{d\varepsilon_{\theta\theta}}{dr} = \frac{d\sigma_{\theta\theta}}{dr} - \nu\left(\frac{d\sigma_{rr}}{dr} + \frac{d\sigma_{zz}}{dr}\right) \tag{1.11a}$$

$$E\frac{d\varepsilon_{zz}}{dr} = \frac{d\sigma_{zz}}{dr} - \nu\left(\frac{d\sigma_{rr}}{dr} + \frac{d\sigma_{\theta\theta}}{dr}\right) = 0 \quad \text{(Plane strain)} \tag{1.11b}$$

Eliminate $d\sigma_{zz}/dr$ from Eq. 1.11a with Eq. 1.11b,

$$E\frac{d\varepsilon_{\theta\theta}}{dr} = (1-\nu^2)\frac{d\sigma_{\theta\theta}}{dr} - \nu(1+\nu)\frac{d\sigma_{rr}}{dr} \tag{1.12}$$

Eq. 1.11b is known by the plane strain condition. Subtract Eq. 1.7b and Eq. 1.7a,

$$E(\varepsilon_{rr} - \varepsilon_{\theta\theta}) = (1+\nu)\big(\sigma_{rr} - \sigma_{\theta\theta}\big) \tag{1.13}$$

Insert the Eqs. 1.12 and 1.13 into Eq. 1.10,

$$(1-\nu)\frac{d\sigma_{\theta\theta}}{dr} - \nu\frac{d\sigma_{rr}}{dr} - \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{1.14}$$

From Eq. 1.6 with $\sigma_{rz} = 0$,

$$\sigma_{\theta\theta} = \sigma_{rr} + r\frac{d\sigma_{rr}}{dr} \qquad \frac{d\sigma_{\theta\theta}}{dr} = \frac{d^2\sigma_{rr}}{dr^2} + 2\frac{d\sigma_{rr}}{dr} \tag{1.15}$$

Substitute Eqs. 1.15 into Eq. 1.14,

$$r\frac{d^2\sigma_{rr}}{dr^2} + 3\frac{d\sigma_{rr}}{dr} = 0$$

Proceed to

$$\frac{d}{dr}\left(r^3\frac{d\sigma_{rr}}{dr}\right) = 0 \tag{1.16}$$

The boundary conditions are given, at $r = R_0$, the outer radius of the cylinder

$$\sigma_{rr}\big(r = R_0\big) = 0$$

at $r = R$, the inner radius of the cylinder

$$\sigma_{rr}(r = R) = -p$$

The solution of Eq. 1.16 is

$$\sigma_{rr} = -p\frac{(R_0/r)^2 - 1}{(R_0/R)^2 - 1} \tag{1.17}$$

With Eq. 1.15,

$$\sigma_{\theta\theta} = p\frac{(R_0/r)^2 + 1}{(R_0/R)^2 - 1} \tag{1.18}$$

With complete axial restraint, the axial strain is zero,

$$\varepsilon_{zz} = 0$$

From Eq. 1.7c,

$$\sigma_{zz} = p\frac{2\nu}{(R_0/R)^2 - 1}$$

which is tensile, independent of r, and smaller than $\sigma_{\theta\theta}$.

When there is no axial stress, then

$$\sigma_{zz} = 0$$

When the axial stress is obtained by equating the force on the inner surface of the upper end $\pi R^2 p$, with the counterbalancing force in the annual cross section, $\pi(R_0^2 - R^2)\sigma_{zz}$.

$$\sigma_{zz} = \frac{p}{(R_0/R)^2 - 1} \tag{1.19}$$

In Eq. 1.19, when the thickness of annulus is very thin

$$\left(\frac{R_0}{R}\right)^2 - 1 = \left(1 + \frac{\delta}{R}\right)^2 - 1 \simeq 1 + 2\left(\frac{\delta}{R}\right) - 1 = \frac{2\delta}{R}$$

Then the stress value reduce to Eq. 1.9. Assume that

$$r = R + \frac{\delta}{2} \rightarrow \frac{R_0}{r} = \frac{R_0/R}{(R + \delta/2)/R} \simeq \frac{R_0}{R}\left(1 - \frac{\delta}{2R}\right)$$

Proceed to

$$\left(\frac{R_0}{r}\right)^2 \simeq \left(\frac{R_0}{R}\right)^2\left(1 - \frac{\delta}{R}\right)$$

Then

$$\frac{(R_0/r)^2 - 1}{(R_0/R)^2 - 1} = 1 - \frac{(R_0/R)^2(\delta/R)}{(R_0/R)^2 - 1} = 1 - \frac{(R_0/R)^2(\delta/R)}{2(\delta/R)} = 1 - \frac{1}{2}\left(\frac{R_0}{R}\right)^2$$

Since

$$\frac{R_0}{R} = 1 + \frac{\delta}{R}$$

For very small δ,

$$\frac{(R_0/r)^2 - 1}{(R_0/R)^2 - 1} \simeq \frac{1}{2}$$

therefore, Eq.1.17 reduce to

$$\sigma_{rr} = -\frac{p}{2}$$

1.7 Spherical shapes

For the internally pressurized spherical shapes, the mathematical manipulations of the elasticity yield

$$\frac{d}{dr}\left(r^4\frac{d\sigma_{rr}}{dr}\right) = 0$$

The solution of ODE is

$$\sigma_{rr} = C_1 + \frac{C_2}{r^3}$$

with boundary conditions

$$\sigma_{rr}(r = R) = -p \qquad \sigma_{rr}\big(r = R_0\big) = 0$$

the radial stress is

$$\sigma_{rr} = -p\frac{\big(R_0/r\big)^3 - 1}{\big(R_0/R\big)^3 - 1} \tag{1.20}$$

the tangential stress is

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = p\frac{\frac{1}{2}\big(R_0/r\big)^3 + 1}{\big(R_0/R\big)^3 - 1} \tag{1.21}$$

In thin-wall limit,

$$R_0 - R = \delta \ll R$$

Eq. 1.21 reduce to

$$\bar{\sigma}_{\theta\theta} = \frac{pR}{2\delta} \tag{1.22}$$

which is half of the hoop stress for the thin-wall cylinder.

1.8 Thermal stress

Incorporating thermal stress,

$$\varepsilon_{xx} = \frac{1}{E}\Big[\sigma_{xx} - \nu\big(\sigma_{yy} + \sigma_{zz}\big)\Big] + \alpha\big(T - T_0\big) \tag{1.23a}$$

$$\varepsilon_{yy} = \frac{1}{E}\Big[\sigma_{yy} - \nu\big(\sigma_{xx} + \sigma_{zz}\big)\Big] + \alpha\big(T - T_0\big) \tag{1.23b}$$

$$\varepsilon_{zz} = \frac{1}{E}\Big[\sigma_{zz} - \nu\big(\sigma_{xx} + \sigma_{yy}\big)\Big] + \alpha\big(T - T_0\big) \tag{1.23c}$$

1.8.1 Axis-symmetric cylindrical geometry

The following analysis applies to inifinitely-long cylindrical annuli as well as to solid cylinders at locations far removed the ends. The temperature is assumed to be a function of radial position only. The surface temperatures are not functions of azimuthal angle θ or axial location z. The thermal expansion coefficient is usually positive, therefore, the hot zone is usually under compression and cold zone is under tensile. Between these two stressed zones lies a surface of a zero stress.

Radial stress

With Eqs. 1.10, 1.14, 1.23a and 1.23b,

$$\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d\sigma_{rr}}{dr}\right) = -\left(\frac{\alpha E}{1-\nu}\right)\frac{1}{r}\frac{dT}{dr} \tag{1.24}$$

This equation is integrated for a long, thick-wall cylinder with inner radius R and outer radius R_o. Pressure loading is considered independently, so the boundary conditions for the thermal-stress state are

$$\sigma_{rr}(r=R) = \sigma_{rr}(r=R_0) = 0$$

The result is

$$\sigma_{rr} = \frac{\alpha E}{1-\nu}\left[\frac{1-R^2/r^2}{R_0^2-R^2}\int_R^{R_0} rT(r)dr - \frac{1}{r^2}\int_R^r r'T(r')dr'\right] \tag{1.25}$$

For a thin-wall cylinder, the approximated solution for temperature profile is obtained by

$$T(r) = T(R) + \frac{\Delta T}{\delta}(r-R) = T(R) + \Delta T\frac{R}{\delta}y \tag{1.26}$$

where $\Delta T = T(R_0) - T(R)$, and

$$y = \frac{r}{R} - 1 \ll 1 \tag{1.27}$$

The thin-wall approximation reduces Eq. 1.25 to

$$\sigma_{rr} = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}y\left(1-\frac{R}{\delta}y\right) \tag{1.28}$$

Plug Eq. 1.25 into Eq. 1.6 without shear stress, we have the hoop stress

$$\sigma_{\theta\theta} = \frac{\alpha E}{1-\nu}\left[\frac{1+R^2/r^2}{R_0^2-R^2}\int_R^{R_0} rT(r)dr - \frac{1}{r^2}\int_R^r r'T(r')dr'\right] \tag{1.29}$$

Apply thin wall approximation in Eqs. 1.26 and 1.27, the hoop stress is

$$\sigma_{\theta\theta} = \frac{1}{2}\Delta T\frac{\alpha E}{1-\nu}\left(1-\frac{2R}{\delta}y\right) \tag{1.30}$$

Axial stress

The axial component of the thermal stress depends on the axial and conditions. The solution method for no axial restraint is given by below.

1. From Eq. 1.23c with $\varepsilon_{zz} = 0$,

$$\sigma'_{zz}(r) = \nu\big(\sigma_{rr} + \sigma_{\theta\theta}\big) - \alpha E\big[T(r) - T(R)\big] \tag{1.31}$$

2. Then cross-section average is

$$\bar{\sigma}'_{zz} = \frac{1}{\pi\left(R_0^2 - R^2\right)}\int_R^{R_0} 2\pi r\sigma'_{zz}(r)dr$$

3. Remove axial constraint.

$$\sigma_{zz}(r) = \sigma'_{zz}(r) - \bar{\sigma}'_{zz}$$

For thin-wall cladding, from Eqs.1.31, 1.28 and 1.30, we have

$$\sigma'_{zz} = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\nu \left[y\left(1 - \frac{R}{\delta}y\right) + 1 - \frac{2R}{\delta}y - 2\frac{1-\nu}{\nu}\frac{R}{\delta}y \right] \simeq A\left(\nu - \frac{2R}{\delta}y\right)$$

where

$$A = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}$$

Change integration variable from r to y,

$$\bar{\sigma}'_{zz} = A\left(\frac{R}{\delta}\right)\int_0^{\delta/R}(1+y)\sigma'_{zz}(y)dy = A\left(\frac{R}{\delta}\right)\int_0^{\delta/R}(1+y)\left(\nu - \frac{2R}{\delta}y\right)dy$$

When $y \simeq 0$,

$$\bar{\sigma}'_{zz} = A\left[\nu\left(1 + \frac{1}{2}\frac{\delta}{R}\right) - 1\right] \simeq -A(1-\nu)$$

Finally,

$$\sigma_{zz} = \sigma'_{zz} - \bar{\sigma}'_{zz} = A\left(1 - \frac{2R}{\delta}y\right) = \frac{1}{2}\Delta T \frac{\alpha E}{1-\nu}\left(1 - \frac{2R}{\delta}y\right) \tag{1.32}$$

1.9 Fuel-pellet cracking due to thermal stresses

Assume the uniform volumetric heating in the solid cylinder cooled to temperature T_s at its periphery (R_0) generates a parabolic temperature distribution

$$\frac{T - T_s}{T_0 - T_s} = 1 - \frac{r^2}{R_0^2} \tag{1.33}$$

and

$$\frac{dT}{dr} = -\frac{2r}{R_0^2}(T_0 - T_s) \tag{1.34}$$

Plug Eq.1.34 into Eq.1.24, then

$$\frac{d}{d\eta}\left(\eta^3 \frac{d\sigma_{rr}}{d\eta}\right) = 8\sigma^* \eta^3$$

where

$$\sigma^* = \frac{\alpha E(T_0 - T_s)}{4(1-\nu)} \qquad \eta = \frac{r}{R_0} \tag{1.35}$$

With the boundary conditions

$$\frac{d\sigma_{rr}}{d\eta} = 0 \qquad \text{when} \qquad \eta = 0$$

$$\sigma_{rr} = 0 \qquad \text{when} \qquad \eta = 1$$

the solution is

$$\sigma_{rr}^{\text{th}} = -\sigma^*(1 - \eta^2) \qquad \sigma_{\theta\theta}^{\text{th}} = -\sigma^*(1 - 3\eta^2) \tag{1.36}$$

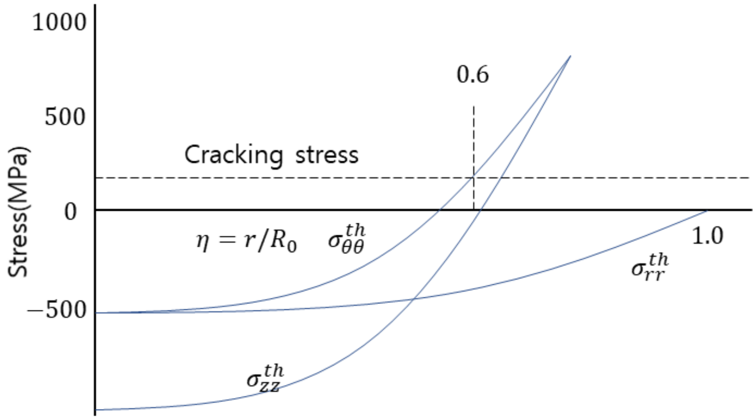

Figure 1.9: Thermal stresses in a fuel pellet under irradiation

The hoop stress is obtained by Eq.1.6 without last term.

Calculating the axial stress distribution is a messy process written on page 393 of "J. H. Rust, *Nuclear Power Plant Engineering* (Atlanta: Haralson, 1979)", leading to the result:

$$\sigma_{zz}^{\mathrm{th}} = -\sigma^*(2 - 4\eta^2) \tag{1.37}$$

The thermal stress solutions satisfy the plane strain condition

$$\frac{d\varepsilon_{zz}}{dr} = 0$$

only near the midplane of the solid cylinder; the ends satisfy the plane stress condition

$$\frac{d\sigma_{zz}}{dr} = 0$$

Assume the situation, a fuel-pellet sustains a centerline-to-surface temperature difference of 530 K. The properties of UO_2 are

1. Young's modulus: $E = 170\,\mathrm{MPa}$
2. Poisson's ration: $\nu = 0.3$
3. Thermal expansion coefficients: $\alpha = 1.5 \times 10^{-5}\,{}^\circ\mathrm{C}^{-1}$

From Eq. 1.35,

$$\sigma^* = \frac{\left(1.5 \times 10^{-5}\right) \times 170 \times 530}{4(1 - 0.3)} = 0.5\,\mathrm{GPa}$$

Fig. 1.9 shows the thermal stress components in the UO_2 pellet. The stress at which UO_2 cracks (that is, the toughness of the fracture, 130 MPa) is shown as the horizontal dashed line in the plot. This stress is exceeded by $\sigma_{\theta\theta}^{\mathrm{th}}$ at a fractional radius of ~ 0.58, at which point radial cracks appear. Similarly, $\sigma_{zz}^{\mathrm{th}}$ becomes greater than the fracture stress at $r/R_0 > 0.75$, beyond which horizontal and vertical cracks extend to the surface of the pellet. The micrographs in Fig. 1.10 show the cracking pattern that results from the stress distribution in Fig. 1.9. Fig. 1.11 illustrates another consequence of the temperature gradients in the irradiated fuel. The *hourglass* shape of the pellet is due to the change from the plane-strain condition near the midplane to the plane-stress condition at the ends. The ends of the pellets contact the cladding, resulting in an external shape that resembles a stalk of bamboo or hourglass.

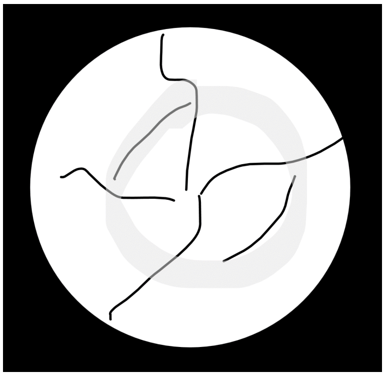

Figure 1.10: Schematic illustration of cracking in a UO_2 fuel pellet.

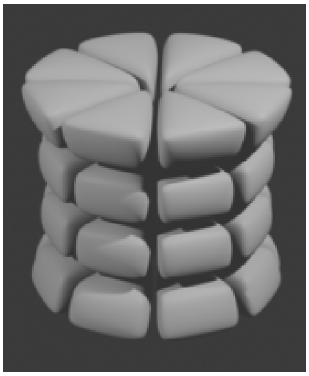

Figure 1.11: Schematic of a fuel pellet with a “hourglassing” shape due tothe effect of thermal stresses

1.9.1 Pellet expansion

The thickness of the gas-filled gap between the fuel pellet and the cladding tube strongly affects the heat transfer between these two components and, consequently, the fuel temperature. The gap thickness decreases from its initial "cold" value as a result of the temperature distribution given by Eq. 1.31 during operation. The reduction in gap thickness is mainly due to thermal expansion of the pellets, although the wedges of the cracked pellets also tend to move out.

When $T = T_s$ and $r = R$ with boundary condition $\sigma_{rr} = 0$, from Eq. 1.23b,

$$\varepsilon_{\theta\theta}(R) = \frac{1}{E}\Big[\sigma_{\theta\theta}(R) - \nu\sigma_{zz}(R)\Big] + \alpha\big(T_s - T_{\mathrm{ref}}\big)$$

where T_{ref} is the pellet temperature in the cold state. The stresses are obtained from Eqs. 1.33, 1.36 and 1.37,

$$\sigma_{\theta\theta}(R) = \sigma_{zz}(R) = \frac{\alpha E(T_0 - T_s)}{2(1-\nu)}$$

Proceed to

$$\varepsilon_{\theta\theta}(R) = \frac{\Delta R}{R} = \alpha(\overline{T} - T_{\mathrm{ref}})$$

where

$$\overline{T} = \frac{1}{2}\big(T_0 + T_s\big)$$

is the average fuel-pellet temperature.

1.9.2 Thermal stress parameter

σ^* of Eq.1.35 is the basis of a measure of the ability of a material to withstand thermal stresses without cracking. It can be generalized by expressing the temperature difference in terms of Fourier's law as

$$\Delta T = \frac{qL}{k}$$

where q is the heat flux, L is the distance over which ΔT occurs, and k is the thermal conductivity. The first two of these parameter are operational, in the sense that they can be changed at will. The thermal conductivity k, however, is a material property. Substituting the Fourier-law expression for ΔT into the definition of σ^*, dividing by the fracture stress σ_{F}, and omitting the product qL leaves a grouping of material properties that serves as a general measure of the thermal-stress resistance of the material:

$$\frac{\sigma_{\mathrm{F}}(1-\nu)k}{\alpha E}$$

called by the thermal stress parameter. Although developed from the solution for a sold cylinder, the concept is applicable to any geometry, with or without heat generation. The larger the thermal stress parameter, the more resistant the solid is to thermal stress failure. The value for Zr cladding is typically $\sim 2\times10^4\,\mathrm{W\,m^{-1}}$ and that for UO_2 is $\sim 200\,\mathrm{W\,m^{-1}}$.